

# Universal fluctuations in KPZ growth on one-dimensional flat substrates

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We present a numerical study of the evolution of height distributions (HDs) obtained in interface growth models belonging to the Kardar-Parisi-Zhang (KPZ) universality class. The growth is done on an initially flat substrate. The HDs obtained for all investigated models are very well fitted by the theoretically predicted Gaussian Orthogonal Ensemble (GOE) distribution. The first cumulant has a shift that vanishes as  $t^{-1/3}$ , while the cumulants of order  $2 \leq n \leq 4$  converge to GOE as  $t^{-2/3}$  or faster, behaviors previously observed in other KPZ systems. These results yield a new evidence for the universality of the GOE distribution in KPZ growth on flat substrates. Finally, we further show that the surfaces are described by the Airy<sub>1</sub> process.

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The last decade witnessed a great advance in the understanding of paradigmatic nonequilibrium models for interface growth by means of rigorous analytical results [1, 2] as well as their experimental realizations [3, 4]. Among the most prominent examples are several models belonging to the Kardar-Parisi-Zhang (KPZ) universality class introduced by the stochastic equation [5]

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \eta, \quad (1)$$

that describes the evolution of an interface  $h(x, t)$ . In this equation,  $\eta$  is a white noise defined by  $\langle \eta \rangle = 0$  and  $\langle \eta(x, t) \eta(x', t') \rangle = D \delta(x - x') \delta(t - t')$ . In  $d = 1 + 1$ , the KPZ equation yields self-affine surfaces with an interface width in a scale of length  $\ell$ , defined as  $w = \sqrt{\langle h^2 \rangle - \langle h \rangle^2}$ , given by the Family-Vicsek scaling ansatz [6]:  $w(\ell, t) = t^\beta \mathcal{F}(\ell/t^{1/z})$ , where  $\mathcal{F}(x) \sim \text{const.}$  for  $x \gg 1$  and  $\mathcal{F}(x) \sim x^{\beta z}$  for  $x \ll 1$ . In  $d = 1 + 1$ , the exponents of the KPZ class are  $\beta = 1/3$  and  $z = 3/2$  [5]. Several models with the scaling exponents of the KPZ universality class have been reported [7, 8].

For system belonging to the KPZ class in  $d = 1 + 1$ , the Family-Vicsek scaling suggests an interface height evolving as [1]

$$h(t) \simeq v_\infty t + \text{sign}(\lambda)(\Gamma t)^{1/3} \chi, \quad (2)$$

where  $v_\infty$  and  $\Gamma$  are non-universal (model dependent) parameters, and  $\chi$  is a time independent random variable. The height distributions (HDs) and consequently  $\chi$  were computed exactly for some models in the KPZ class (see Refs. [1, 2] for recent reviews), strongly suggesting that  $\chi$  is a universal feature of the KPZ class. In a pioneer work, Johansson [9] established a link between random matrix theory and KPZ class by determining the exact asymptotic HDs of the totally asymmetric exclusion process

(single step model) as the Tracy-Widom (TW) distribution of the Gaussian unitary ensemble (GUE) [10]. In the same year, Prähofer and Spohn [11, 12] determined the HDs of the radial polynuclear growth (PNG) model as a GUE distribution whereas the Gaussian orthogonal ensemble (GOE) distribution was obtained for the growth on a flat substrate. Quite recently, exact solutions of the KPZ equation in  $d = 1 + 1$  corroborated GOE distributions for flat [13] and GUE for radial [14, 15] geometries, respectively.

Besides TW distributions for the heights, the limiting processes describing the surfaces were identified as the so-called Airy processes [1]. For circular growth, analytic solutions [16], experiments [3], and simulations [17] strongly suggest that the limiting process ruling the KPZ universality class is given by the dynamics of the largest eigenvalue in Dyson's Brownian motion of GUE matrices, the Airy<sub>2</sub> process. Analogously, a limiting process for the flat case, named Airy<sub>1</sub>, was found for PNG [18] and a similar model [19], both belonging to the KPZ class, indicating that it may be a universal feature of the KPZ class. It is worth mentioning that Airy<sub>1</sub> process does not correspond to the dynamics of largest eigenvalue in Dyson's Brownian motion of GOE matrices [20].

KPZ universality class in 1+1 dimensions was convincingly verified in a meticulous experiment involving two turbulent phases in the electroconvection of nematic liquid crystal films [3, 4]. Using an unprecedented statistics for experimental systems ( $\sim 10^3$  realizations), it was possible to observe not only the KPZ scaling exponents  $\beta = 1/3$  and  $z = 3/2$ , but also the GUE and GOE distributions for radial and flat geometries, respectively. In both cases, cumulants of order  $n = 2, 3$  and 4 converge fast to the corresponding GOE and GUE cumulants, but the mean ( $n = 1$ ) approaches the theoretical value as a power law  $t^{-1/3}$ . This decay was also found in an analytical solution of the KPZ equation in  $d = 1 + 1$  with an edge initial condition [14] and in simulations of radial Eden model [17]. However, Ferrari and Frings [21] have shown the non-universality of the amplitudes of this correction and that higher order cumulants have no correction up to order  $\mathcal{O}(t^{-2/3})$ .

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model	$v_\infty$	$\Gamma$	$A$
BD	2.1398(7)	4.90(5)	2.7(2)
RSOS	0.41903(3)	0.252(1)	0.81(3)
KPZ I	0.05725(4)	0.00081(1)	0.008(1)
KPZ II	0.08130(5)	0.00493(2)	0.035(2)

TABLE I: Non-universal parameters for the investigated models.

The analysis of height distributions in computer simulations of non-solvable models, supposedly belonging to the KPZ class, are relatively less frequent than the analytical counterpart of solvable models. Indeed, some reports for flat geometries dating from the beginning of nineties, when computer resources were quite limited, demonstrate a surprisingly good agreement with the lately found out theoretical GOE distribution [22, 23]. For a radial geometry, we have recently observed asymptotic GUE distributions in computer simulations of distinct Eden models grown from a single seed [17].

In the present work, we revisit the computer simulations of non-solvable models having the KPZ exponents in  $1 + 1$  dimensions. A flat initial condition is used. We also integrated the KPZ equation numerically. In all cases, we confirmed that HDs and cumulants agree with the GOE distribution. As recently observed for other systems [3, 4, 14], a correction in the mean approaching zero as  $t^{-1/3}$  was found while corrections in higher order cumulants consistent with  $t^{-2/3}$  were obtained for three of four investigated models. These results are in agreement with the corrections obtained analytically for solvable KPZ models [21]. Finally, the covariance of all investigated models agrees with the  $\text{Airy}_1$  process, as verified for some solvable models [18, 19].

We study the restricted solid-on-solid (RSOS) and ballistic deposition (BD) models [7] as well as the numerical integration of Eq. (1) for two sets of parameters. In all cases, we use one-dimensional lattices of size  $L = 2^{20}$  and periodic boundary conditions. Statistical averages were performed over at least 400 independent samples. The RSOS model consists in the deposition of a particle in a randomly chosen site. The deposition is rejected everytime the height difference between nearest neighbors is greater than one lattice unity. In the BD model, the deposition site is randomly chosen and the particle falls normally to the substrate. When the first contact with the deposit happens, it irreversibly sticks in this position.

We investigated a discrete one-dimensional KPZ equation given by [24]

$$h_j(t + \Delta t) - h_j(t) = \frac{\Delta t}{(\Delta x)^2} \{ \nu(h_{j-1} - 2h_j + h_{j+1}) + \frac{\lambda}{8}(h_{j-1} - h_{j+1})^2 \} + \sigma \sqrt{12\Delta t} R, \quad (3)$$

where  $\sigma \equiv \sqrt{2D/\Delta x}$  and  $R$  is a random number uni-

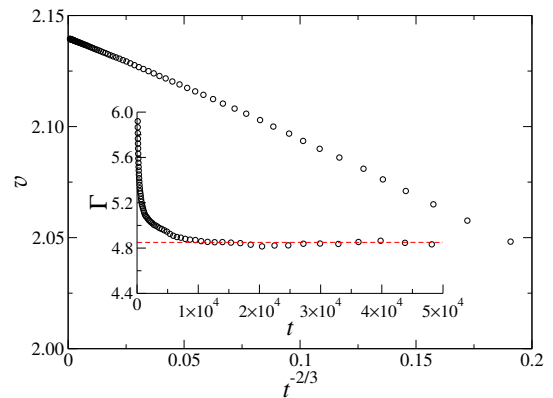


FIG. 1: (Color online) Growth velocity  $v$  against  $t^{-2/3}$  for the BD model. Inset shows  $\Gamma$  against time for BD. Dashed line represents the estimate shown in Table I.

formly distributed in the interval  $[-0.5, 0.5]$ . Integration instabilities are prevented by replacing the gradient term in KPZ equation by  $f(x) = (1 - e^{-C|\nabla h|^2})/C$  where  $C$  is a parameter to be adjusted to control the instability [25]. Integration was performed for fixed parameters  $\Delta x = 1$ ,  $\Delta t = 0.04$  and  $\nu = 0.5$ . Two sets of parameters were studied. In the first one, named as KPZ I, we used  $\sigma = 0.1$ , the coupling constant  $g \equiv \lambda^2 D / \nu^3 = 24$ , and  $C = 0.5$ . In the second one, the KPZ II, we used  $\sigma = 0.2$ ,  $g = 12$  and  $C = 0.2$ .

Comparison between HDs and random matrix distributions needs highly accurate estimates of the parameters  $v_\infty$  and  $\Gamma$  given by Eq. (2). In BD and RSOS models studied here, estimates are reported in literature [22], but the accuracy is not sufficient for the aims of the present work. Thus, we determined the parameters for all studied models. Using Eq. (2), the asymptotic growth velocity can be obtained from the extrapolation of plots  $v \equiv d\langle h \rangle / dt$  against  $t^{-2/3}$  when  $t \rightarrow \infty$ , as illustrated in Fig. 1 for the BD model. The asymptotic velocities for all investigated models are presented in Table I. Notice that our estimates for RSOS and BD models are consistent with those formerly reported in Ref. [22]:  $v_\infty = 0.419$  and  $2.14$ . The parameter  $\Gamma$  was obtained from the second order cumulant of the HDs. Accordingly Eq. (2), it reads as  $\langle h^2 \rangle_c \simeq (\Gamma t)^{2/3} \langle \chi^2 \rangle_c$ , where  $\langle x^n \rangle_c$  is the  $n$ th cumulant of  $x$ . Since we expect that  $\chi$  fluctuations are given by the GOE distribution, we used  $\langle \chi^2 \rangle_c = \langle \chi_{GOE}^2 \rangle_c = 0.63805$  [11] in our calculations. The insertion to Fig. 1 shows  $\Gamma \simeq (\langle h^2 \rangle_c / \langle \chi_{GOE}^2 \rangle_c)^{3/2} / t$  as a function of time for the BD model. The estimated  $\Gamma$  values for all investigated models are presented in Table I. Plots similar to Fig. 1 were obtained for all models.

We study the rescaled HDs following the KPZ ansatz given by Eq. (2). If this equation holds, the fluctuations of  $q \equiv (h - v_\infty t) / (\Gamma t)^{1/3}$  would be given by the GOE distribution. Fig. 2 shows the rescaled distributions  $P(q)$  for all investigated models. As expected, the agreement with GOE is noticeable, but the distributions are slightly

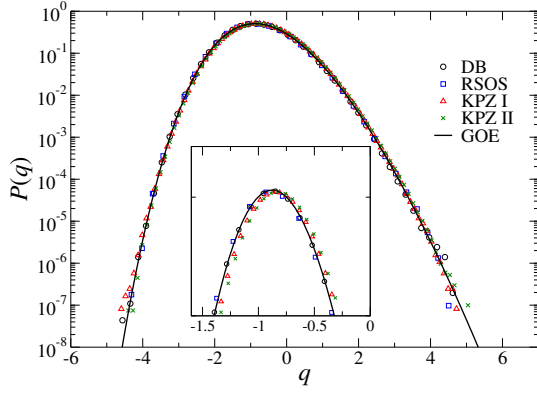


FIG. 2: (Color online) Height distributions scaled accordingly Eq. (2) for all investigated models. Simulation time of  $t = 2 \times 10^4$  was used for BD and RSOS and  $t = 800$  for the integrated KPZ equation. The solid line is the GOE distribution. The inset shows a zoom around the peak of the distributions.

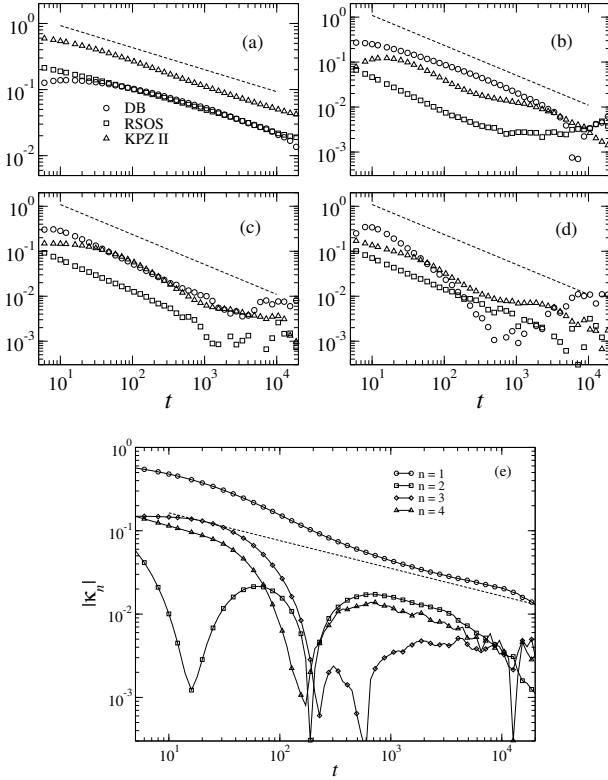


FIG. 3: Absolute differences between cumulants of the scaled HDs and GOE distribution against time. Results for (a) first, (b) second, (c) third, and (d) fourth order cumulants are shown. Results for KPZ I are shown apart in (e). The dashed lines have slopes  $-1/3$  in (a) and (e) and  $-2/3$  in (b)-(d).

shifted in relation to GOE, as highlighted in the inset of Fig. 2.

The absolute differences between  $P(q)$  and GOE cumulants,  $|\kappa_n|$ , are shown as functions of time in Fig. 3. The first cumulant difference decays closely to a power law  $t^{-1/3}$  for all investigated models, except for KPZ I.

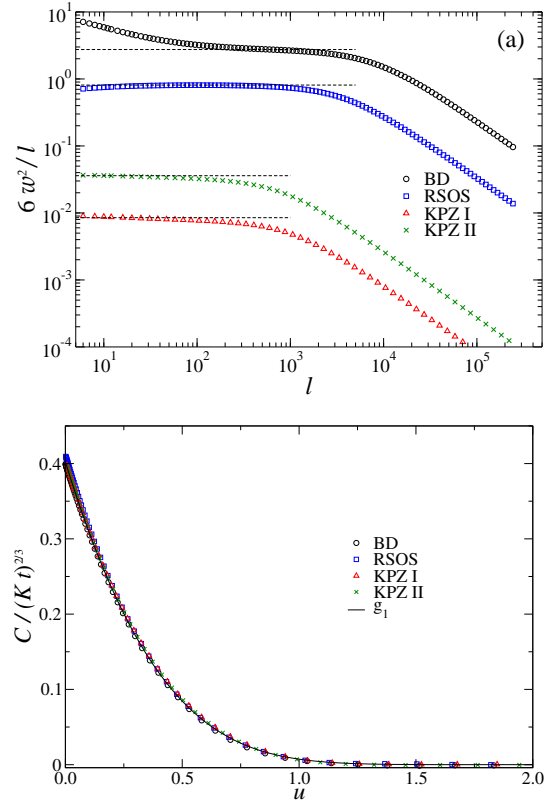


FIG. 4: (Color online) (a) Rescaled local roughness for different models. Simulation times are  $t = 5 \times 10^4$  for BD and RSOS and  $t = 2 \times 10^3$  for KPZ I and II. (b) Scaled two-point correlation function against  $u \equiv (Al/2)/(Kt)^{2/3}$  for different models. Time is  $t = 10^4$  for BD and RSOS and  $t = 400$  for the integrated KPZ equation. The solid line is the covariance of the Airy<sub>1</sub> process.

In the latter, the initial decay is faster but seems to converge to  $t^{-1/3}$ . Ferrari *et al.* [21] have shown that the corrections in the first cumulant for a number of solvable models belonging to the KPZ class read as

$$\kappa_1 = \langle q \rangle - \langle \chi \rangle = at^{-1/3} + \mathcal{O}(t^{-2/3}) \quad (4)$$

where  $a$  is a model dependent constant that can even be null for some models. Numerically, if the factor  $a$  is much smaller than the factor of higher order correction, one must observe a fast initial decay [ $\mathcal{O}(t^{-2/3})$ ] followed by a  $t^{-1/3}$  decay. This is observed in the KPZ I results shown in Fig. 3(e). For KPZ I, the higher order cumulants reach GOE very fast and then the differences  $|\kappa_n|$  fluctuate around zero. Furthermore, the non-universality of the constant  $a$  is verified in our simulation since  $a$  is negative for BD and RSOS and positive for the KPZ equation integration. Ferrari *et al.* [21] have also shown that the higher order cumulants have no correction up to order  $\mathcal{O}(t^{-2/3})$ . Our simulations are in agreement with this claim as one can see in Figs. 3(b)-3(e), where some curves are consistent with a  $t^{-2/3}$  decay while others decay faster, which is particularly evident in KPZ I case.

The conclusion of Ref. [21] stating the non-universality of the correction refers only to the amplitudes since the scenario suggested by Eq. (4) is quite general: any correction to the first moment, if it exists, should decay with an exponent  $-1/3$ , while corrections in higher order cumulants, if they exist, must decay as fast as or faster than  $t^{-2/3}$ .

Besides the universality of the height fluctuations, the limiting process describing the interface is an important issue on surface growth. To check if our simulations yield surfaces given by an  $\text{Airy}_1$  processes, we studied the two-point correlation function

$$C(l, t) \equiv \langle h(x+l, t)h(x, t) \rangle - \langle h \rangle^2, \quad (5)$$

that is expected to scale as  $C(l, t) \simeq (Kt)^{2/3}g_1(u)$  with  $u \equiv (Al/2)/(Kt)^{2/3}$ , where  $g_1(u)$  is the covariance of the  $\text{Airy}_1$  process and  $K = \lambda A^2 = 2\Gamma$  [16]. The amplitude  $A$  can be obtained from the squared local roughness

$$w^2(l, t) = \langle h^2(x, t) \rangle_l - \langle h(x, t) \rangle_l^2 \simeq Al/6, \quad (6)$$

where  $\langle \dots \rangle_l$  means averages over windows of size  $l$  [3, 22]. Fig. 4(a) shows the plots used to determine the amplitudes shown in Table I. The short plateau for the BD model is due to strong finite time effects of this model. Notice that the amplitude is given by  $A = D/\nu$  for the KPZ equation, implying that  $A = 0.01$  and  $A = 0.04$  are expected for KPZ I and II, respectively. The slightly different amplitudes observed are due to the approximated integration method [26]. Figure 4(b) shows the results

for the rescaled two-point correlation function exhibiting a very satisfactory agreement among all models and the covariance of the  $\text{Airy}_1$  process ( $g_1$ ). This confirms that all models are described by the  $\text{Airy}_1$  process and gives one more evidence of this universal feature in KPZ universality class. Similar results were obtained at different times, indicating small finite-time effects in the two-point correlation function.

In conclusion, we investigated non-solvable models in the KPZ universality class in one-dimensional flat substrates as well as a numerical integration of the KPZ equation. We analyzed the height distributions and the two-point correlation function. Height fluctuations of all investigated models are described by GOE distributions as conjectured for the KPZ class for flat substrates, and previously observed in analytical [13] and experimental [4] works. The cumulant analysis yields a correction vanishing as  $t^{-1/3}$  for the first cumulant whereas the corrections for higher order cumulants go to zero as  $t^{-2/3}$  or faster. The latter result is in agreement with theoretical analysis of Ferrari *et al* [21]. Finally, we have found that the limiting processes describing the surface is the  $\text{Airy}_1$  process in agreement with previous results for solvable models [18, 19].

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